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СТРУКТУРНО-ДИНАМИЧЕСКАЯ ЭКОНОМИЧЕСКАЯ МОДЕЛЬ SDEM-2: ОТ СИСТЕМНО-ДИНАМИЧЕСКИХ РЕШЕНИЙ К МАКСИМИЗАЦИИ ЛИНЕЙНОЙ И ЛОГАРИФМИЧЕСКОЙ ПОЛЕЗНОСТИ

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Структурно-динамическая экономическая модель SDEM-2 является моделью закрытой экономики, растущей в условиях конфликта интересов двух влиятельных агрегированных экономических акторов: предпринимателей и наемных работников. Экономический рост в модели SDEM-2 исследуется в рамках как системно-динамического, так и оптимизационного подходов. В рамках системно-динамического подхода рассматриваются четыре альтернативные управляющие стратегии предпринимателей: «альтруистическая», «умеренный рост выпуска», «здесь и сейчас» и «умеренный рост дивиденда». При оптимизационном подходе с использованием принципа максимума Понтрягина решаются задачи максимизации линейной и логарифмической полезности. Оценивается степень субоптимальности системно-динамических решений

Ключевые слова: ЭКОНОМИЧЕСКИЙ РОСТ, ДИНАМИЧЕСКАЯ СИСТЕМА, ДИНАМИЧЕСКАЯ ОПТИМИЗАЦИЯ

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THE STRUCTURAL DYNAMIC ECONOMIC MODEL SDEM-2: FROM SYSTEM DYNAMIC SOLUTIONS TO LINEAR AND LOGARITHMIC UTILITY MAXIMIZATION

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The Structural Dynamic Economic Model SDEM-2 is essentially a model of a closed economy growing under conditions of conflict of interests of two powerful aggregate actors: entrepreneurs and wage-earners. We study the economic growth within SDEM-2 both in system dynamic and optimization model setups. For the system dynamic model setup, four alternative control strategies of entrepreneurs are considered in detail: the “altruistic” control strategy, the “moderate output growth” control strategy, the “here and now” control strategy, and the “moderate dividend growth” control strategy. In the optimization setup the Pontryagin's maximum principle is applied to SDEM-2 to solve the linear and logarithmic utility maximization problems. The degree of sub-optimality of system dynamic solutions is evaluated

Keywords: ECONOMIC GROWTH, DYNAMIC SYSTEM, DYNAMIC OPTIMIZATION

1. Introduction

The Structural Dynamic Economic Model (SDEM) was initially proposed in [2] and later served as a basis for the development of the Multi-Actor Dynamic Integrated Assessment Model (MADIAM) aimed at modeling the dynamics of the coupled climate-socioeconomic system under conditions of global warming [5, 10] – a problem broadly discussed in the economics of climate change [8-9]. In the current paper we provide a brief description of an upgraded model SDEM-2 following Ref. [7] and study the dynamics of a closed economy driven by conflict of interests of two model actors: employers *(entrepreneurs)* and employees *(wage-earners).* We treat SDEM-2 both in the system dynamic and the optimization setups.

The paper is organized as follows. Sec. 2 is devoted to a brief description of the model SDEM-2. Sec. 3 describes the system-dynamic model setup. Four alternative control strategies of entrepreneurs are considered in detail: the “altruistic” strategy, the “moderate output growth” strategy, the “here and now” strategy, and the “moderate dividend growth” strategy. In Sec. 4 the dynamic optimization techniques based on the Pontryagin's maximum principle are applied to SDEM-2 to maximize linear and logarithmic utility. Sec. 5 concludes.

2. SDEM-2: model description

In this section we provide a brief overview of the model SDEM-2: detailed derivations of model equations can be found in [6, 7].

SDEM-2 is a model of a closed economy. The population dynamics is exogenous (the exponential growth of population with a constant rate λ_L is assumed), and population (equal to labor force as full employment is assumed) is divided into two social classes: entrepreneurs comprising a constant fraction θ of population ($0<\theta<1$) and wage-earners comprising the remainder $(1-\theta)$ of population.

The structure of consumption in the economy is twofold. Wage-earners consume everything they earn (the average wages of a wage-earner are denoted as w). Entrepreneurs do not earn wages – instead, each entrepreneur consumes the “dividend” d .

The per capita output y in the economy depends on per capita physical capital k and per capita human capital h as two primary production factors. Unlike in standard economic growth models (cf. [1]), these two forms of capital are assumed to be non-substitutable. For the corresponding production function we adopt the Leontief form

$$y = \min(\nu k, \mu h) \tag{1}$$

where ν, μ are constant factors. An in-depth discussion of the background behind Eq. (1) can be found in [4-6]. In what follows we consider a regime of balanced growth when

$$\nu k = \mu h \,. \tag{2}$$

The dynamics of the economy is governed by a system of three first-order ordinary differential equations:

$$\dot{k} = i_k - (\lambda_k + \lambda_L)k \,, \tag{3}$$

$$\dot{h} = i_h - (\lambda_h + \lambda_L)h \,, \tag{4}$$

$$\dot{w} = \lambda_w (q w_{\text{targ}} - w) \,. \tag{5}$$

Eqs. (3)-(4) are capital dynamics equations, i_k and i_h being the investments in physical and human capital, respectively, and λ_k and λ_h being the depreciation rates of physical and human capital, respectively. Eq. (5) is the wage dynamic equation describing the wage negotiation process between trade unions of wage-earners and entrepreneurs: q is a parameter reflecting the negotiation power of entrepreneurs ($0 < q < 1$), λ_w is the wage adjustment rate, and $w_{\text{targ}}(t)$ is a (changing) *target wage rate* dependent on the current state of the economy (for the discussion on choice of investments i_k and i_h and calculation of $w_{\text{targ}}(t)$, see [4-6]).

We now come to non-dimensional variables. First, we introduce an auxiliary constant

$$\varpi_0 = \mu\nu - \mu\lambda_k - \nu\lambda_h - (\mu + \nu)\lambda_L \tag{6}$$

and a non-dimensional parameter

$$\gamma = \frac{\varpi_0}{\lambda_w(\mu + \nu)} \,. \tag{7}$$

We note then that if we adopt the method of quantifying the macroeconomic parameters in goods units (as opposed to monetary units), then [k]=good, [w]=[d]=good/year. We introduce new variables with tildes

$$\tilde{w} = \frac{(1-\theta)\mu}{\varpi_0} w, \tag{8}$$

$$\tilde{d} = \frac{\theta\mu}{\varpi_0} d, \tag{9}$$

and a non-dimensional time

$$\tau = \lambda_w t. \tag{10}$$

Note that k, $\tilde{w}$ and $\tilde{d}$ can be regarded as non-dimensional, if a non-dimensional value is assigned to the unit "good".

Finally, the problem (3)-(5) can be posed in a non-dimensional form (tildes have been removed from non-dimensional variables)

$$\dot{k} = \gamma(k - w - d), \tag{11}$$

$$\dot{w} = qk - w, \tag{12}$$

$$0 \le d \le k - w, \tag{13}$$

$$0 < w_0 \le k_0. \tag{14}$$

Note that the dimensionality of the dynamic system has been reduced (from 3D to 2D) in view of the additional condition of balanced growth (Eq. (2), see [6-7] for a discussion). The second inequality in Eq. (13) follows from the additional constraint $\dot{k} \ge 0$. Eq. (14) is a constraint on initial conditions to make them compatible with the non-zero dividend at the initial time.

To close the model, we should somehow specify the entrepreneurs' dividend d – the control variable of the problem. We will use two alternative model setups: *the system-dynamic model setup* and *the optimization setup.*

In *system-dynamic model setup* entrepreneurs choose $d(\tau)$ at every instant according to a certain formalized control strategy, either in the form

$$d(\tau) = D_1(k(\tau), w(\tau)) \tag{15}$$

(as adopted in [5, 10]); or, alternatively, in the form

$$\dot{d}(\tau) = D_2\left(k(\tau), w(\tau), d(\tau)\right) \tag{16}$$

(as discussed in detail in [4, 6]).

The optimization model setup implies maximization of entrepreneurs' utilities. We will use either linear or logarithmic utility function, calculating the utilities in non-dimensional form:

$$U_*^{\text{lin}} = \int_0^\infty d(\tau) \exp(-\Delta\tau) d\tau\,; \tag{17}$$

$$U_*^{\log} = \int_0^\infty \ln[d(\tau)] \exp(-\Delta\tau) d\tau\,, \tag{18}$$

where

$$\Delta = \frac{\delta}{\lambda_w} \tag{19}$$

is a non-dimensional discount rate (δ is the dimensional discount rate in Eq. (19)).

3. System-dynamic setup

3.1. The matrix form of dynamic equations

In this section we treat SDEM-2 in system-dynamic model setup. This means that the system (11)-(14) should be supplemented by some control strategy of entrepreneurs formalizing their dividend choice. In what follows we will consider several control strategies of the form given by Eq. (15).

It is convenient to rewrite the dynamic equations (11)-(12) in the matrix form. To do this, we introduce the state vector

$$\mathbf{x} = (k, \quad w)^{\mathrm{T}} \tag{20}$$

with the initial condition

$$\mathbf{x}(0) = \mathbf{x}_0 = (k_0, \quad w_0)^{\mathrm{T}}, \tag{21}$$

the matrix

$$\mathbf{A} = \begin{pmatrix} \gamma, & -\gamma \\ q, & -1 \end{pmatrix}, \tag{22}$$

and an auxiliary vector

$$\mathbf{e}_1 = (1, \quad 0)^{\mathrm{T}}. \tag{23}$$

Then Eqs. (11)-(14) can be rewritten in a compact from

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \gamma \mathbf{e}_1 d(\tau), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{24}$$

$$0 < w_0 \le k_0, \tag{25}$$

$$0 \le d \le k - w. \tag{26}$$

In what follows we will consider four different entrepreneurs' control strategies to close the system (24)-(26): the "altruistic" strategy, the "moderate output growth" strategy, the "here and now" strategy, and the "moderate dividend growth" strategy.

3.2. Control strategies of entrepreneurs

3.2.1. "Altruistic" control strategy

We consider first an (unrealistic) case in which altruistic business strives only to maximize the growth rate of the economy by maximizing the investment, and takes no dividend:

$$d(\tau) = 0. \tag{27}$$

The treatment of this case has little economic sense in itself, however, the results of this subsection will be broadly used in the remainder of the paper.

The dynamic equation (24) then takes the homogeneous form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{28}$$

with the symbolic solution

$$\mathbf{x}(\tau) = \exp(\mathbf{A}\tau)\mathbf{x}_0. \tag{29}$$

To examine the dynamic properties of the solution we have to calculate the eigenvalues of the matrix $\mathbf{A}$.

The eigenvalues $\lambda_{\pm}$ obey the characteristic equation

$$D_{\mathbf{A}}(\lambda) \equiv \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad (30)$$

where $\mathbf{I}$ is a unit matrix. If $q < 1$, then the eigenvalues have opposite signs, and the following inequality holds:

$$-1 < \lambda_{-} < 0 < \lambda_{+} < \gamma . \quad (31)$$

An example of the numeric solution for “altruistic” control strategy is shown on Fig. 1.

3.2.2. “Moderate output growth” control strategy

We now assume that entrepreneurs choose their dividend in such a way that the steady growth of per capita physical capital (and consequently of per capita output) at a constant rate α is maintained. Essentially, this means that the output evolves like in the standard *AK* model [1]. The dynamic system (24)-(26) yields in this case:

$$k(\tau) = k_0 \exp(\alpha\tau) , \quad (32)$$

$$w(\tau) = \frac{qk_0}{1+\alpha} \exp(\alpha\tau) + \left(w_0 - \frac{qk_0}{1+\alpha} \right) \exp(-\tau) , \quad (33)$$

$$d(\tau) = -\frac{D_A(\alpha)}{\gamma(1+\alpha)} k_0 \exp(\alpha\tau) - \left(w_0 - \frac{qk_0}{1+\alpha} \right) \exp(-\tau) \quad (34)$$

(in the last equation we have used the notation from Eq. (30)). Since for $0 < \alpha < \lambda_{+}$ we have $D_{\mathbf{A}}(\alpha) < 0$ then in this range of α we obtain that $d(\tau) \to +\infty$ when $\tau \to +\infty$.

Note that the control strategy (34) can be equivalently presented in a form of a simple rule

$$d(\tau) = \left(1 - \frac{\alpha}{\gamma} \right) k(\tau) - w(\tau) . \quad (35)$$

To sum up, in case of the “moderate output growth” strategy the per capita capital grows exponentially at the rate α (though less rapidly than when the “altruistic” strategy is adopted). The wage rate and the dividend also grow exponentially in the asymptotic limit at the same rate α.

3.2.3. “Here and now” control strategy

Let us consider now the “moderate output growth” strategy (see Sec. 3.2.2) in the limiting case when $\alpha = 0$, i.e. when the per capita capital does not grow at all:

$$k(\tau) = \text{const} = k_0 . \tag{36}$$

This situation corresponds to the case in which entrepreneurs decide to choose at every instant the maximum dividend possible for the current state of the economy (the “here and now” strategy):

$$\dot{k} = 0 , \qquad d(\tau) = k(\tau) - w(\tau) . \tag{37}$$

The dynamic system (24)-(26) yields in this case:

$$w(\tau) = qk_0 + (w_0 - qk_0)\exp(-\tau) , \tag{38}$$

$$d(\tau) = (1-q)k_0 - (w_0 - qk_0)\exp(-\tau) . \tag{39}$$

Obviously, the “here and now” strategy leads to stagnation: neither the per capita capital nor the output grow, while the wage rate and dividend converge asymptotically to constant values. However, as we will see in Sec. 4.1, in the linear utility maximization model setup and for the case of a sufficiently large discount rate, self-interested entrepreneurs striving to maximize their utility will indeed choose this “here and now” strategy leading to the socially non-optimal situation of stagnation.

The numeric solution for “here and now” control strategy is shown on Fig. 2.

3.2.4. “Moderate dividend growth” control strategy

Finally we consider a strategy when entrepreneurs are willing to maintain the growth rate of *dividend* (not of *output*, like it was in Sec. 3.2.2) at a constant rate α:

$$d(\tau) = d_0 \exp(\alpha\tau) , \tag{40}$$

where the initial value of the dividend d_0 has to be chosen at a maximum level still obeying the model constraints.

Substituting Eq. (40) into Eq. (24), we easily find the solution in the symbolic form

$$\mathbf{x}(\tau) = \exp(\mathbf{A}\tau)\left(\mathbf{x}_0 - (\mathbf{A}-\alpha\mathbf{I})^{-1}\mathbf{e}_1\gamma d_0\right) + (\mathbf{A}-\alpha\mathbf{I})^{-1}\mathbf{e}_1\gamma d_0 \exp(\alpha\tau)\,. \tag{41}$$

Clearly, the bigger the initial value of the dividend d_0 the better for entrepreneurs. However, the solution should obey the proper constraints. It can be shown that entrepreneurs can choose d_0 in Eq. (40) according to the formula

$$d_0 = \min\left(\frac{\lambda_+ - \alpha}{\gamma}\left[k_0 - \frac{\gamma}{\gamma - \lambda_-} w_0\right], k_0 - w_0\right). \tag{42}$$

We will see below that in the case of logarithmic utility maximization (Sec. 4.2) the exponential growth of dividend (i.e. the “moderate dividend growth” control strategy) proves to be the optimal entrepreneurs’ choice.

4. Optimization setup

4.1. Linear utility maximization

Now we come to the optimization model setup for SDEM-2. As discussed in Sec. 2, the dynamic equations and constraints should be supplemented in this case by the utility maximization condition. In this subsection we consider the dynamic system (24)-(26) supplemented by the linear utility maximization condition (20):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \gamma\mathbf{e}_1 d(\tau)\,, \qquad \mathbf{x}(0) = \mathbf{x}_0\,, \tag{43}$$

$$0 < w_0 \le k_0\,, \tag{44}$$

$$0 \le d \le k - w\,, \tag{45}$$

$$\max U_*^{\text{lin}} \equiv \max \int_0^\infty d(\tau)\exp(-\Delta\tau)\,d\tau\,. \tag{46}$$

This problem can be solved by applying the Pontryagin's maximum principle in its general form [3] allowing to take into account the constraints on control variables depending on state variables (such a constraint appears in the second inequality in Eq. (45)).

For the linear utility maximization problem under consideration the corresponding Hamiltonian is linear in d, therefore it is evident that the system always evolves along one of the boundaries of the phase space:

$$\text{either } d = 0 \text{ or } d = k - w, \tag{47}$$

with possible occasional instant discontinuous “switches” from one boundary to the other (the relay control). Therefore we immediately choose the “here and now” entrepreneurs’ control strategy when $d(\tau) = k(\tau) - w(\tau)$ as a candidate for the optimal solution. But would not the entrepreneurs be willing to stick at some time spans to the “altruistic” control strategy ($d(\tau) = 0$) and to sacrifice their dividend for a while, allowing the economy to grow as fast as it can, and then switching to the “here-and-now” strategy to enjoy the extensive consumption with more attractive initial conditions?

An analysis shows that if $\Delta > \lambda_+$, then the “here and now” control strategy of entrepreneurs (Sec. 3.2.3) is the optimal strategy when the linear utility (46) is maximized. In the opposite case $\Delta < \lambda_+$ the linear utility (46) can be made arbitrarily high by combining the “altruistic” and “here and now” control strategies.

A constructive example of a set of control strategies providing the arbitrarily high level of linear utility when $\Delta < \lambda_+$ is as follows: initially to adopt the “altruistic” control strategy ($d(\tau) \equiv 0$, $\tau < \tau_1$) and then to switch to “here-and-now” strategy ($d(\tau) = k(\tau) - w(\tau)$, $\tau > \tau_1$). It can be shown that $U_*^{\text{lin}}(\tau_1) \to +\infty$ when $\tau_1 \to +\infty$.

Explicitly, the maximum value of utility (46) when $\Delta > \lambda_+$ and entrepreneurs adopt the ``here-and-now'' (HN) strategy, can be calculated from a simple formula

$$U_{\text{HN}*}^{\text{lin}} = \mathbf{f}_{\text{HN}}^{\text{T}} \mathbf{x}_0, \tag{48}$$

where

$$\mathbf{f}_{\mathrm{HN}} = \left(\frac{1-q+\Delta}{\Delta(\Delta+1)}, \quad -\frac{1}{\Delta+1} \right)^{\mathrm{T}}, \tag{49}$$

and $\mathbf{x}_0$ is the initial condition.

It is instructive to calculate the linear utilities for three other control strategies considered in Sec. 3.2 and to evaluate the degree of "sub-optimality" of these control strategies. We omit the derivations and provide here only the final results.

(i) The "altruistic" control strategy (A):

$$U_{\mathrm{A}*}^{\mathrm{lin}} = 0; \tag{50}$$

(ii) The "moderate output growth" control strategy (MOG):

$$U_{\mathrm{MOG}*}^{\mathrm{lin}} = \mathbf{f}_{\mathrm{MOG}}^{\mathrm{T}} \mathbf{x}_0, \tag{51}$$

where

$$\mathbf{f}_{\mathrm{MOG}} = \left(-\frac{D_{\mathbf{A}}(\alpha)}{\gamma(1+\alpha)} + \frac{q}{1+\alpha} \cdot \frac{1}{\Delta+1}, \quad -\frac{1}{\Delta+1} \right)^{\mathrm{T}}; \tag{52}$$

(iii) The "moderate dividend growth" control strategy (MDG):

$$U_{\mathrm{MDG}*}^{\mathrm{lin}} = \mathbf{f}_{\mathrm{MDG}}^{\mathrm{T}} \mathbf{x}_0, \tag{53}$$

where, depending on the relation between k_0 and w_0, either

$$\mathbf{f}_{\mathrm{MDG}} = \frac{1}{\gamma} \frac{\lambda_+ - \alpha}{\Delta - \alpha} \cdot \left(1, \quad -\frac{\gamma}{\gamma - \lambda_-} \right)^{\mathrm{T}} \tag{54}$$

or

$$\mathbf{f}_{\mathrm{MDG}} = \frac{1}{\Delta - \alpha} \cdot (1, \quad -1)^{\mathrm{T}}. \tag{55}$$

The degree of sub-optimality η_i of i-th control strategy from (i)-(iii) can then be evaluated according to the formula

$$\eta_i = 1 - \frac{U_{i*}^{\mathrm{lin}}}{U_{\mathrm{HN}*}^{\mathrm{lin}}}. \tag{56}$$

4.2. Logarithmic utility maximization

Finally we consider the logarithmic utility maximization problem. The system of equations (43)-(45) should be supplemented now by the logarithmic utility maximization condition (23):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \gamma \mathbf{e}_1 d(\tau), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{57}$$

$$0 < w_0 \le k_0, \tag{58}$$

$$0 \le d \le k - w, \tag{59}$$

$$\max U_*^{\log} \equiv \max \int_0^\infty \ln[d(\tau)] \exp(-\Delta\tau) d\tau. \tag{60}$$

We write down the Hamiltonian

$$H = \ln[d] \exp(-\Delta\tau) + \boldsymbol{\varphi}^{\mathrm{T}} (\mathbf{A}\mathbf{x} - \gamma \mathbf{e}_1 d), \tag{61}$$

where $\boldsymbol{\varphi} = (\varphi_k, \ \ \varphi_w)^{\mathrm{T}}$ is a vector of shadow variables. The maximization of H in d provides the condition

$$d = \frac{\exp(-\Delta\tau)}{\gamma \varphi_k}, \tag{62}$$

while the dynamic system for shadow variables takes the explicit form

$$\dot{\boldsymbol{\varphi}} = -\mathbf{A}^{\mathrm{T}} \boldsymbol{\varphi}. \tag{63}$$

It is easy to check that if the eigenvalues of the matrix $\mathbf{A}$ are equal to λ_+, λ_-, and $\lambda_- < 0 < \lambda_+$, then the eigenvalues of the matrix ($-\mathbf{A}^{\mathrm{T}}$) are equal to ($-\lambda_+$), ($-\lambda_-$), and $-\lambda_+ < 0 < -\lambda_-$. Therefore the general solution of the homogeneous system (63) is given by a formula

$$\boldsymbol{\varphi}(\tau) = \boldsymbol{\varphi}_+ \exp(-\lambda_+ \tau) + \boldsymbol{\varphi}_- \exp(-\lambda_- \tau) \tag{64}$$

where $\boldsymbol{\varphi}_+$, $\boldsymbol{\varphi}_-$ are eigenvectors of the matrix ($-\mathbf{A}^{\mathrm{T}}$).

The transversality condition ($\boldsymbol{\varphi}(\tau) \to 0$ when $\tau \to +\infty$) forces us to exclude the second (growing) term from the r.h.s. of Eq. (64). Thus,

$$\boldsymbol{\varphi}(\tau) = \boldsymbol{\varphi}_+ \exp(-\lambda_+ \tau), \tag{65}$$

and, particularly,

$$\varphi_k(\tau) = \varphi_{k0} \exp(-\lambda_+ \tau). \tag{66}$$

By substituting Eq. (66) into Eq. (62) we get a candidate for the optimal solution in the form of the "moderate dividend growth" control strategy with $\alpha = \lambda_+ - \Delta$:

$$d(\tau) = d_0 \exp\left((\lambda_+ - \Delta)\tau\right). \tag{67}$$

Restricting ourselves to the practically interesting case of growing, not decaying, economy, we impose the constraint $\Delta < \lambda_+$. Then we can make use of the results of Sec. 3.2.4 to write down the optimal solution. Having calculated in such a way the initial value d_0, we are then able to calculate the utility (60), the final result being

$$U_{\max,*}^{\log} = \frac{1}{\Delta}\left(\ln d_0 + \frac{\lambda_+}{\Delta} - 1\right). \tag{68}$$

In the following we will restrict ourselves to the case $w_0 < qk_0$. By substituting $\alpha = \lambda_+ - \Delta$ into Eq. (42) we get

$$d_0 = \min\left(\frac{\Delta}{\gamma}\left[k_0 - \frac{\gamma}{1+\lambda_+} w_0\right], k_0 - w_0\right) \tag{69}$$

(we have used here the identity $\gamma - \lambda_- = 1 + \lambda_+$).

It can be shown that for $w_0 < qk_0$ the first option in Eq. (69) is always smaller than the second one, so

$$d_0 = \frac{\Delta}{\gamma}\left[k_0 - \frac{\gamma}{1+\lambda_+} w_0\right]. \tag{70}$$

Therefore we come to the optimal strategy of the form

$$d(\tau) = \frac{\Delta}{\gamma}\left[k_0 - \frac{\gamma}{1+\lambda_+} w_0\right] \exp\left((\lambda_+ - \Delta)\tau\right). \tag{71}$$

It can be easily shown that in case of optimal control strategy the identity (70) holds not only for the initial time, but also for every time:

$$d(\tau) = \frac{\Delta}{\gamma}\left[k(\tau) - \frac{\gamma}{1+\lambda_+} w(\tau)\right]. \tag{72}$$

So the entrepreneurs actually have to follow a very simple rule (72) when choosing the value of the dividend.

The sub-optimality of other control strategies can be evaluated in this case by applying a formula analogous to Eq. (56).

5. Conclusions

The model SDEM-2 considered in this paper proved to be quite unique in that it overcomes in many respects the watershed between system-dynamic macroeconomic models and optimization models (the latter being more deeply implanted in neoclassical economic growth theory [1]). Indeed, as is shown in Sec. 4, two of the four quite natural and straightforward control strategies of entrepreneurs – the “here and now” control strategy (Sec. 3.2.3) and the “moderate dividend growth” control strategy (Sec. 3.2.4) – provide at the same time the optimal solutions in case of linear and logarithmic utility maximization problems, respectively. From purely mathematical viewpoint, one of the reasons for this “coincidence” is in adoption of the Leontief production function (Eq. (1)) which is essentially linear in physical (and human) capital in case of balanced growth (Eq. (2)) on which the current paper is focused.

We are planning to use the results obtained in this paper when developing a stylized Integrated Assessment model based on system-dynamic version of SDEM-2 to evaluate the efficiency of potential imposition of global carbon price as climate mitigation policy [9-10].

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FIGURES

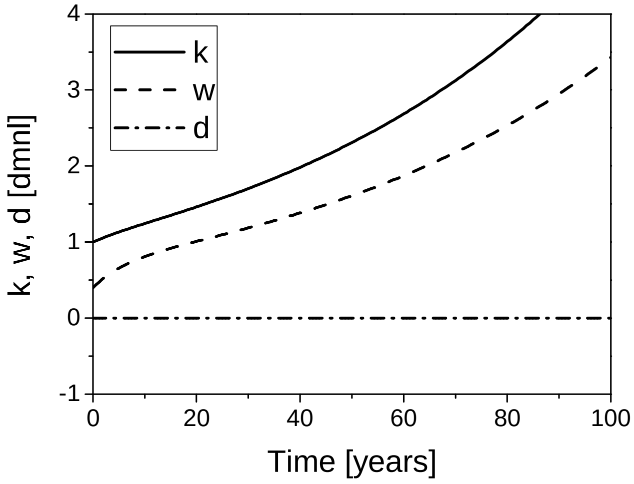

Figure 1: The dynamics of non-dimensional capital k, wages w and dividend d for "altruistic" control strategy.

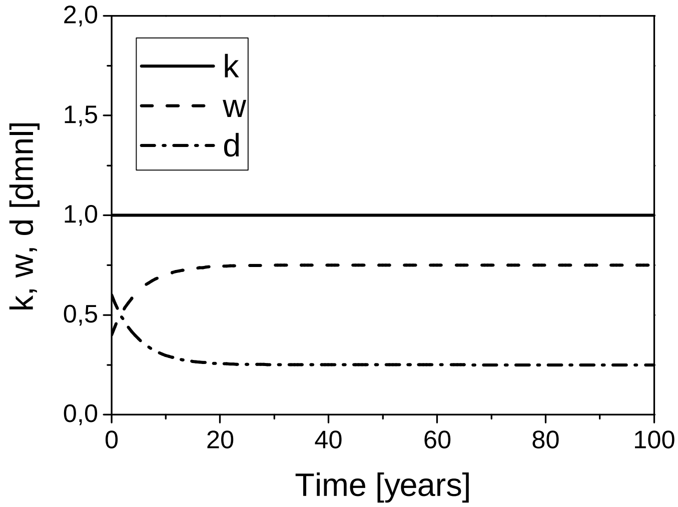

Figure 2: The dynamics of non-dimensional capital k, wages w and dividend d for "here and now" control strategy.