УДК 514.84

UDC 514.84

ГРАВИТАЦИОННЫЕ ВОЛНЫ И КВАНТОВАЯ ТЕОРИЯ

GRAVITATIONAL WAVES AND QUANTUM THEORY

Трунев Александр Петрович
к.ф.-м.н., Ph.D.
Директор, A&E Trounev IT Consulting, Торонто, Канада

Alexander Trunev
Cand.Phys.-Math.Sci., Ph.D.
Director, A&E Trounev IT Consulting, Toronto, Canada

В работе рассмотрена теория гравитации в многомерных пространствах. Сформулирована модель метрики, удовлетворяющая основным требованиям квантовой теории. Показано, что в такой метрике гравитационные волны описываются уравнением Лиувилля. Доказана гипотеза Шредингера о связи волновой функции с гравитационными волнами

In this article we consider gravitation theory in multidimensional space. The model of the metric satisfying the basic requirements of quantum theory is proposed. It is shown that gravitational waves are described by the Liouville equation. Schrödinger conjecture about the Schrödinger wave function and gravitational waves has been proved

Ключевые слова: ГРАВИТАЦИОННЫЕ ВОЛНЫ, КВАНТОВАЯ ТЕОРИЯ, ТЕОРИЯ СТРУН, ТЕОРИЯ ГРАВИТАЦИИ ЭЙНШТЕЙНА, ТЕМНАЯ МАТЕРИЯ, ТЕМНАЯ ЭНЕРГИЯ

Keywords: GENERAL RELATIVITY, GRAVITATIONAL WAVES, BLACK ENERGY, BLACK MATTER, STRING-THEORY

Introduction

General relativity is the basic theory of modern cosmology and quantum gravity [1-13]. Einstein's gravitational field equations have the form [1]:

$$R_{mn} - \frac{1}{2} g_{mn} R = g_{mn} \Lambda + \frac{8pG}{c^4} T_{mn} \tag{1}$$

R_{mn}, g_{mn}, T_{mn} - The Ricci tensor, the metric tensor and energy-momentum tensor; Λ, G, c - Einstein's cosmological constant, the gravitational constant and the speed of light, respectively.

In general, the affine connection and Ricci tensor are

$$\begin{aligned} & R_{ik} = R^j_{ijk}, \quad R = g^{ik} R_{ik}, \\ & R^a_{bgd} = \frac{\partial \Gamma^a_{bd}}{\partial x^g} - \frac{\partial \Gamma^a_{bg}}{\partial x^d} + \Gamma^m_{bd}\Gamma^a_{mg} - \Gamma^m_{bg}\Gamma^a_{md}, \\ & \Gamma^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right) \end{aligned} \tag{2}$$

R^a_{bgd} - Riemann tensor, Γ^i_{kl} - Christoffel symbols of the second kind.

There are several problems concerning energy-momentum tensor [2] and cosmological constant [3-6]. Energy-momentum tensor of matter in the equation (1), in general, depends on the gravitational field. In this regard, Einstein and Infeld [2] formulated program: "All attempts to represent matter by an energy-momentum tensor are unsatisfactory and we wish to free our theory from any particular choice of such a tensor. Therefore we shall deal here only with gravitational equations in empty space, and matter will be represented by singularities of the gravitational field."

This approach to the problem of the origin of matter is not unique. To keep the basic idea of the metric in the Einstein gravitation theory, we assume that the Einstein equation (1) splits into two independent equations [7-8]:

$$\begin{aligned} &R_{mn} - \frac{1}{2} g_{mn} R = k g_{mn} \\ &\frac{8pG}{c^4} T_{mn} = g_{mn}(k - \Lambda) \end{aligned} \qquad (3)$$

Here k - a function depending on the space dimension. Note that the first equation is determined by the space-time, and the second equation is given by the distribution of matter, which corresponds to this metric. This hypothesis is consistent with the idea of the origin of matter from the gravitational field [2, 9-11] but without specific assumptions about the presence of the singularity of the metric.

In our paper [7] model (3) was used to construct a metric of inhomogeneous rotating universe. A mechanism for the production of matter from dark energy by the phase transition has been proposed. In paper [8] we presented model of quantum gravity in space of dimension D with metric

$$\begin{aligned} &ds^2 = y(t,r)dt^2 - p(y)dr^2 - df_1{}^2 - \sin^2 f_1 df_2{}^2 - \sin^2 f_1 \sin^2 f_2 df_3{}^2 - \\ &... - \sin^2 f_1 \sin^2 f_2 ... \sin^2 f_{N-1} df_N{}^2 \end{aligned} \qquad (4)$$

Here $f_1, f_2, ..., f_N$ - angles on the unit sphere, immersed in $D-1$ space. Metric (4) describes many important cases of symmetry used in elementary particle physics and the theory of Supergravity [13-15]. This approach allows capturing the diversity of matter, which produces factory of nature, by choosing the equation of state $p = p(y)$.

In this paper, the metric (4) and model (3) used to justify the Schrödinger hypothesis [16] on the relationship of gravitational waves with the wave function of quantum mechanics.

Supergravity and the motion of matter

Consider gravity in space with the metric (4). Einstein's equation in the form (3) is universal, so it can be generalized to any Riemann space. We will describe the motion of matter by Hamilton-Jacobi equation, which also can be generalized to any Riemann space. Together, these two equations constitute a universal model describing the motion of matter in D-space:

$$R_{mn} - \frac{1}{2} g_{mn} R = k g_{mn} \tag{5}$$

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} = 0 \tag{6}$$

The field equations in the metric (4) are reduced to one second-order equation [8]

$$-p' y_{tt} + y_{rr} = -Kpy - \frac{pp' - 2p'' py + p'^2 y}{2py} y_t^2 + \frac{p + p' y}{2py} y_r^2 \tag{7}$$

In general, the parameters of the model and the scalar curvature depends only on the space dimension, we have

$$\begin{aligned} &k = D(D-5)/2 + 3, \\ &K = 2(D-3), \\ &R = -D^2 + 3D \end{aligned} \tag{8}$$

Note that equation (7) changes its type depending on the sign of the derivative p':

For $p' < 0$ equation (7) is of elliptic type;

For $p' > 0$ equation (7) is of hyperbolic type;

For $p' = 0$ equation (7) has a parabolic type.

Signature of the metric (4) does not change if we require further $p(y) > 0, y > 0$.

Hamilton-Jacobi equation in the metric (4) has the form

$$\frac{1}{y}\left(\frac{\partial S}{\partial t}\right)^2 - \frac{1}{p}\left(\frac{\partial S}{\partial r}\right)^2 - \left(\frac{\partial S}{\partial f_1}\right)^2 - \sin^{-2} f_1\left(\frac{\partial S}{\partial f_2}\right)^2 - \\ \sin^{-2} f_1 \sin^{-2} f_2\left(\frac{\partial S}{\partial f_3}\right)^2 - ... - \sin^{-2} f_1 \sin^{-2} f_2 ... \sin^{-2} f_{N-1}\left(\frac{\partial S}{\partial f_N}\right)^2 = 0 \quad (9)$$

Equation (9) can be integrated, under certain assumptions, using a method that offered Schrodinger [16]. The essence of the method is to represent the solution of equation (11) in the form

$$S = S_{cl} + \mathbf{h} \ln \Psi_S \quad (10)$$

Here explicitly introduced the classical action - S_{cl}, Planck's constant and the wave function Ψ_S. Using the classical action, we define the parameters of the problem, which may be external to the quantum system. With this approach becomes apparent association of quantum mechanics with classical mechanics [17].

In the case of the metric (4) it is convenient to choose as variables of quantum mechanics angles on the unit sphere, and as the variables of the classical action - time and radial coordinate. So, equation (9) is divided into two equations

$$\frac{1}{y}\left(\frac{\partial S_{cl}}{\partial t}\right)^2 - \frac{1}{p}\left(\frac{\partial S_{cl}}{\partial r}\right)^2 = M^2 \\ \left(\frac{\partial \Psi_S}{\partial f_1}\right)^2 + \sin^{-2} f_1\left(\frac{\partial \Psi_S}{\partial f_2}\right)^2 + ... + \sin^{-2} f_1 \sin^{-2} f_2 ... \sin^{-2} f_{N-1}\left(\frac{\partial \Psi_S}{\partial f_N}\right)^2 = \frac{M^2}{\mathbf{h}^2}\Psi_S^2 \quad (11)$$

Here M - an arbitrary constant.

Gravitational waves

Consider gravitational waves that occur in the metric (4) in the case of linear state equation. Put in equation (7)

$$p = y / c^2, y = e^w .$$

Then equation (7) reduces to the Liouville equation:

$$w_{tt} = c^2 w_{rr} + Ke^w \tag{12}$$

For equation (12), we can specify an algorithm for constructing the general solution and particular solutions of different types of solitary waves [18-24]. The general solution is given by (Liouville, 1853)

$$w(r,t) = \ln\left[\frac{8c^2 f'(h) g'(V)}{K\left(f(h) + g(V)\right)^2}\right], \quad h = ct - r, V = r + ct \tag{13}$$

Here $f(h), g(V)$ - arbitrary functions. Note that equation (12) is widely used in string theory and quantum gravity [21-24], since the corresponding model is completely integrable. Usually this equation is derived from the principle of stationary action [21-24], but our method has the advantage that it is possible to define a metric corresponding to gravitational waves and the motion of test particles in this metric.

Indeed using Liouville formula (13), we can specify the general solution of the Einstein equations in the form (3), which describes the gravitational waves in the metric (4):

$$\begin{gathered} y(r,t) = \frac{8c^2 f_h(h) g_V(V)}{K\left(f(h) + g(V)\right)^2}, \quad p(y) = y / c^2, \\ K = 2(D-3), \quad h = ct - r, V = r + ct \end{gathered} \tag{14}$$

Gravitational waves of the type (14) are distributed in a combination comprising advanced and retarded waves. This fact suggests that the scalar gravitational waves may serve as a source of quantum motion of particles, for example, in the form of de Broglie waves [25].

Indeed, we write the first equation (11) in the metric (14), we have

$$\frac{1}{c^2}\left(\frac{\partial S_{cl}}{\partial t}\right)^2 - \left(\frac{\partial S_{cl}}{\partial r}\right)^2 = \frac{8M^2}{K}\frac{f_h(h)g_V(V)}{\left(f(h)+g(V)\right)^2} \tag{15}$$

Assuming that the effect depends on the coordinates h, V, we transform both sides of equation (15) to the equivalent form

$$4\left(\frac{\partial S_{cl}}{\partial h}\right)\left(\frac{\partial S_{cl}}{\partial V}\right) = \frac{8M^2}{K}\frac{\partial \ln\left(f(h)+g(V)\right)}{\partial h}\frac{\partial \ln\left(f(h)+g(V)\right)}{\partial V} \tag{16}$$

It follows from eq. (16) that the action can be expressed in terms of arbitrary functions $f(h), g(V)$ in the form

$$S_{cl} = M\sqrt{\frac{2}{K}}\ln[f(h)+g(V)] \tag{17}$$

Equation (17) can be seen in the opposite direction, suggesting that the unknown functions $f(h), g(V)$ related to the action of test particles

$$f(h)+g(V) = \exp(S_{cl}/h),\ h = M\sqrt{\frac{2}{K}} \tag{18}$$

If we assume that $p = -y/c^2$, then, setting in equation (7) $y = e^w$, we arrive to the Liouville equation of elliptic type

$$w_{tt} + c^2 w_{rr} = Ke^w \tag{19}$$

In this case, we can also get the solutions of equation (19) of the general form, which can be expressed in terms of analytic functions [19]. Application of the elliptical model (32) in the quantum theory of gravity can be found in [22].

Equation (18) allows us to define a metric if arbitrary motion of test particles known, whereas equation (17) to determine the action, if known metric. Similar connection between an action and a wave function we have in the Schrödinger quantum theory [16]. Therefore it can be assumed that the gravitational waves are associated with waves of matter, the existence of which was first pointed out de Broglie [25].

In this case, equation (17) is a direct proof of the de Broglie hypothesis of matter waves, and equation (18) is proof of the Schrödinger hypothesis [16], which was based on de Broglie's and Einstein ideas to create the quantum mechanics.

Matter waves, predicted by de Broglie in 1924, were found experimentally as early as 1927, when Davisson and Germer carried out the electron diffraction on a crystal of nickel in Bell Lab [26]. However, the search for gravitational waves predicted by Einstein in 1916 [1] was unsuccessful up to now.

In this regard, let me excerpt from the letter Schrödinger to Einstein: "... I have been thinking that to be identified ψ-wave with the breaking waves of the gravitational potential - certainly not those that you studied for the first time, but with those that have real mass, i.e. T_{ik} not disappeared. This means, I think we need to abstract general relativity containing T_{ik} even as «asylum ignorantial» (by your own words), to introduce the matter is not as massive points or anything like that, but as a quantized gravitational waves" [16].

If gravitational waves associated with matter waves, as suggested by Schrödinger, there must be some effects due to the emission and absorption of gravitational waves by electrons in atoms. However, until recently, no such effects were detected. The theory of gravitational radiation [27-29], developed for the hydrogen atom, yields a transition rate of

$$\Gamma(3d \to 1s) = \frac{a^6 G m_e^3 c}{360\,\mathbf{h}^2} = 5.7 \cdot 10^{-40}\, s^{-1} \tag{20}$$

Here $a = \dfrac{e^2}{\mathbf{h}c}$ - the fine structure constant.

As is known, in Einstein's theory [1] the gravitational radiation is associated with the quadrupole moment of the mechanical system [1, 27-29]. However, from equation (18) it follows that the gravitational waves associated with action.

Note that Einstein created the theory of gravitational radiation in the 1918 [1], never imagined that his theory of gravity will be used in the quantum theory without any improvement. Moreover, Einstein believed that the development of quantum theory will create a theory of gravity that is free from the problem of thermal graviton radiation. Einstein assumed that the intensity of the emission of gravitational waves from the moving matter associated with the quadrupole moment, but such a way that the system does not lose energy long enough.

In this sense, the expression (20) reflects the semiclassical approach in quantum gravity, which does not take into account the true nature of gravitational waves and the quantum nature of gravity transitions. The result is an extremely low rate (20), so the effect of gravitational radiation apparently cannot be observed [27].

On the other hand, the de Broglie waves were registered not only in laboratories but also widely used in applied physics. However, due to certain historical reasons, these waves are not identified with gravitational waves. Development of quantum mechanics went in the direction opposite to that indicated by Schrödinger. Only recently, due to the discovery of dark energy and dark matter we have the problem of baryonic matter, the contents of which in the universe is no more than 5 % [12, 30].

If the content of baryonic matter is so small, why particles observed in the laboratories are only connected with baryonic matter, but there is no trace of dark energy or dark matter? This question, in our opinion, is directly related to gravitational waves, which have also not been found, despite all efforts. It is possible that the answer lies on the surface with one of interpretation of equations

(17) - (18), which indicates a direct link de Broglie waves and Schrödinger wave function with gravitational waves.

Schwarzschild metric and the state equation

All static metrics (4) described by equation (7), assuming that in this equation $y_t = y_{tt} = 0$, we find

$$y_{rr} = -Kpy + \frac{p + p'y}{2py} y_r^2 \tag{21}$$

Integrating eq. (21), we obtain

$$py(C - 2Ky) = y_r^2 \tag{22}$$

C - Arbitrary constant. For physical applications there are static solutions which have the asymptotic like the Schwarzschild metric

$$y = 1 - \frac{2m}{R} \tag{23}$$

Metric (23) is widely used in the theory to the phenomenon of collapse, leading to the formation of black holes [27-28, 31-32]. Note that the Schwarzschild metric is defined in spherical coordinates, while the metric (4) is centrally symmetric. To match metrics we set $R = 1/r$, then the Schwarzschild metric is transformed to

$$y = 1 - 2mr \tag{24}$$

An example of static metric with asymptotic as the Schwarzschild metric (23) is the exponential metric

$$y = \exp(-2mr) = 1 - 2mr + ... \tag{25}$$

Substituting (25) into equation (22), we find the state equation, which is in according with the Schwarzschild metric

$$p = \frac{4m^2 y}{C - 2Ky} = \frac{2m^2}{K} \frac{-1}{1 \pm \exp(2mr - m)},$$
$$\frac{dp}{dy} = \frac{C}{(C - 2Ky)^2}, \quad C = \pm 2Ke^{-m} \qquad (26)$$

Note that in the Schwarzschild metric (23) parameter m corresponds to the mass and the rest energy of the system. We assume that in the metric (4) the point mass source of gravity is due to two types of equations of state corresponding to “bosons” and “fermions”.

Let compare the first equation (26) with the quantum statistics:

- In the case of “bosons” $p = \frac{2m^2}{K} \frac{1}{\exp(2mr - m) - 1}, \quad C > 0, p(y) > 0, p'(y) > 0$;

- In the case of “fermions” $p = -\frac{2m^2}{K} \frac{1}{\exp(2mr - m) + 1}, \quad C < 0, p(y) < 0, p'(y) < 0$.

This division of matter on the “bosons” and “fermions” with using the state equation (26) is conditional. As we know, in statistical physics, by definition, fermions and bosons are particles which obey quantum statics at first. The spin-statistics connection depending on wave function symmetry was proved by Pauli in his famous theorem [33]. But there is no obligation that any bosons or fermions have spins and wave functions as it requires in the Pauli theorem. This kind of particles we call “bosons” and “fermions”.

Existence of matter in the form of “fermions” leads to the fact that the field equation (7) is of elliptic type. As is known, in this case, small perturbations grow exponentially, which is consistent with the general behavior of the baryonic matter in the universe, subject to inflation [5].

On the other hand, the existence of matter in the form of “bosons” leads to the field equation of hyperbolic type. In such a space - time there are gravitational waves of finite amplitude. In the case of linear perturbations of the state equation we have $p = y / c^2$, therefore these waves can be described by wave solutions of the Liouville equation (12).

Solitary waves

Besides gravitational waves there is free movement at a constant speed, which can be determined by requiring the equation (7) the presence of solutions of the form $y = y(r + ut)$, then (7) reduces to an ordinary differential equation

$$(1 - u^2 p')y'' = -Kpy - \frac{pp' - 2p''py + p'^2 y}{2py} u^2 y'^2 + \frac{p + p'y}{2py} y'^2 \qquad (27)$$

In the particular case, choosing the equation of state in the form $p = ay$, find the general solution of the equation (27)

$$y = 2\frac{(1 - au^2)}{aK} b^2 \left(1 - \tanh^2\left[b(r + ut - r_0)\right]\right) \qquad (28)$$

Here b, r_0 - arbitrary constants. The resulting solution describes a solitary wave that propagates at a constant speed in the radial direction – Figure 1.

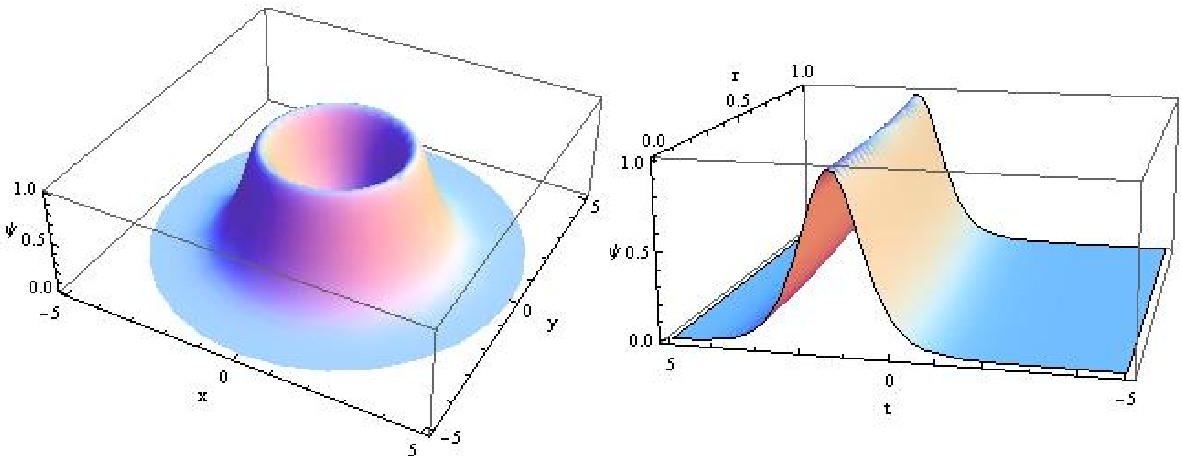

Figure 1: Gravitational solitary wave.

If we assume that $p = y \ / c^2$, and we require that at the origin satisfy the equation $y \ = 1$, then the metric (28) in the case of four-dimensional space-time takes the form like in Lorentz's theory:

$$y \ = 1 - \tanh^2\left(\frac{r + ut - r_0}{\sqrt{c^2 - u^2}}\right) \quad (29)$$

In a case of the state equation (26) we can solve eq. (27) numerically with initial condition $y \ (0) = y_{\ 0}, y \ '(0) = 0$ and for different value of radial velocity in a range $0 \le u \le 0.9$ - Figure 2.

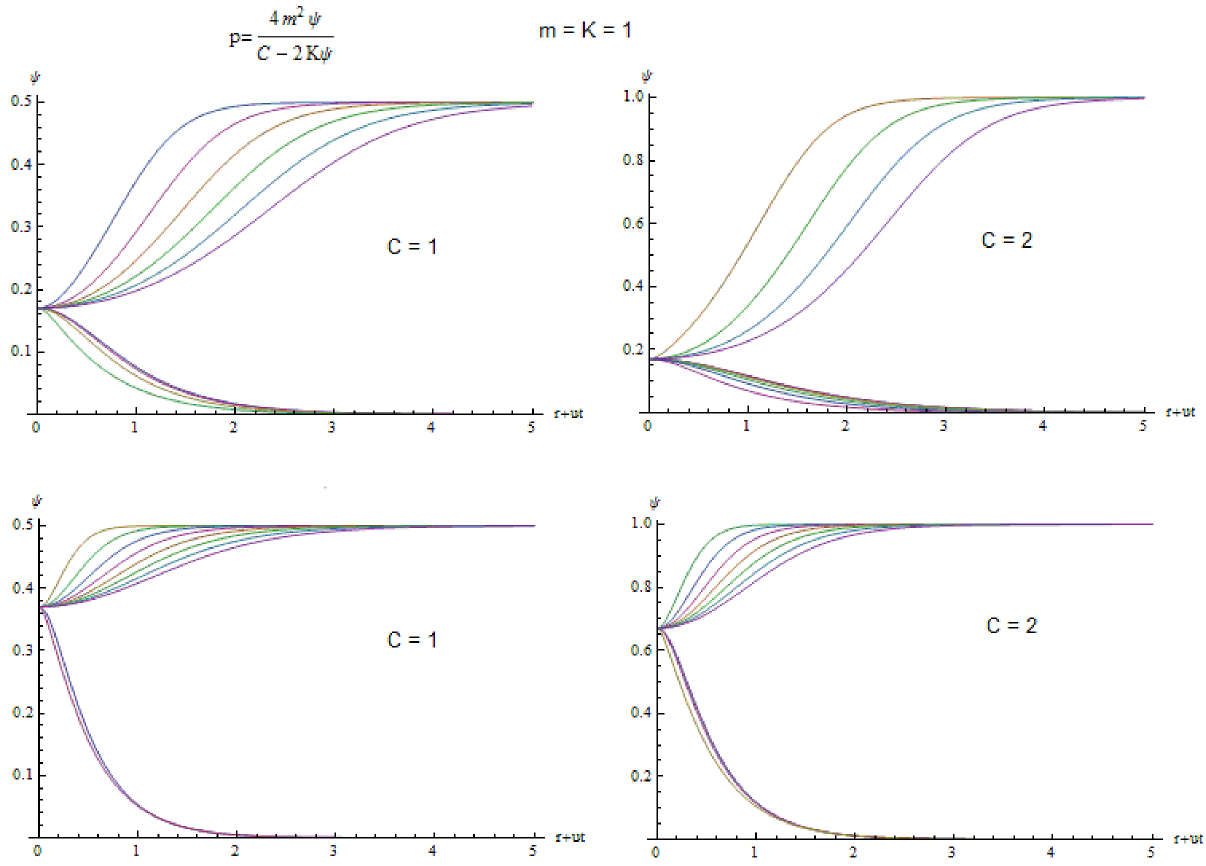

Figure 2: Metrics of “boson” system calculated on eq. (26)-(27) for $m = K = 1; C = 1;2;\ 0 \le u \le 0.9$. Radial velocity is increasing from bottom to top curves.

As it mentioned above system of “bosons” is stable in a wide range of radial velocity. There are two types of metric related to the Schwarzschild metric (23) and geon metric [9-10]. Both of them are depending on radial velocity, initial conditions and the state equation. For small value of the radial velocity we have Schwarzschild metric (23), and for relatively higher – geon metrics (see Figure 2).

Numerical solutions in a form of gravitational wave and geon were found together for different value of radial velocity in a range from 0.1 to 0.9 and for the state equation $p(y) = \dfrac{1}{C+y}$ - Figure 3.

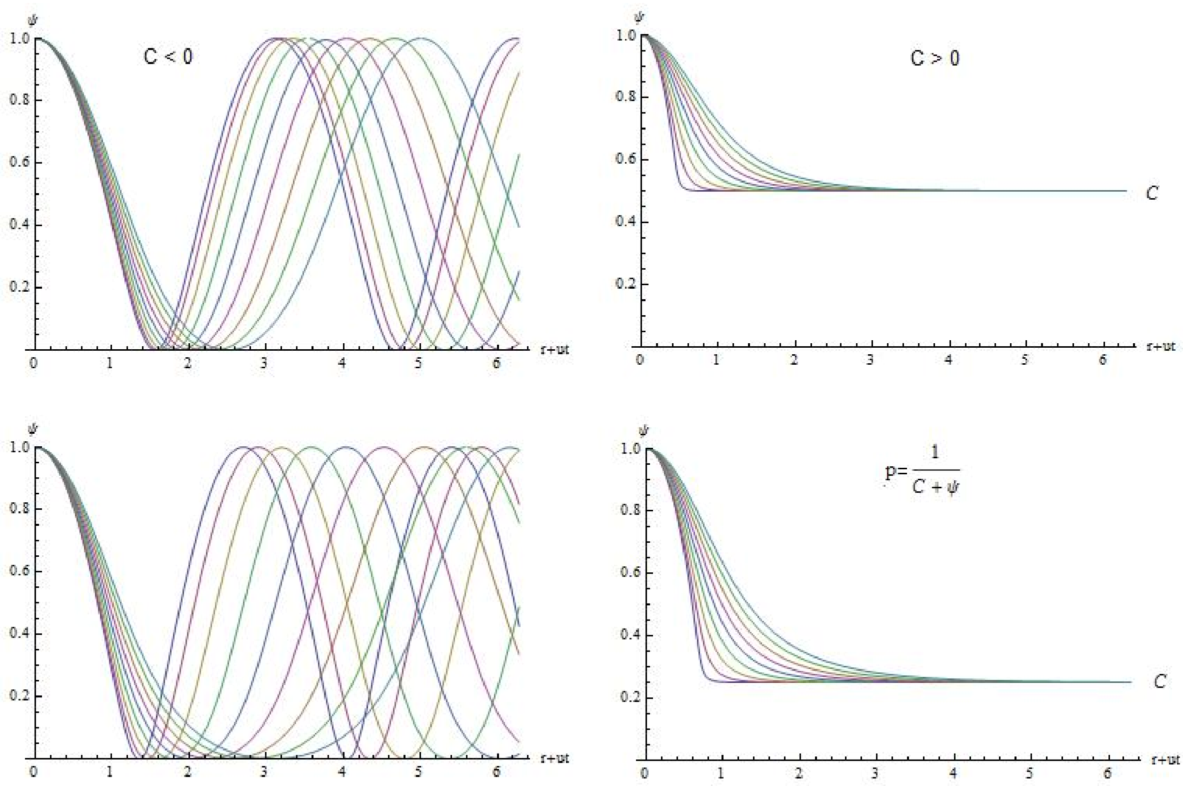

Figure 3: Metrics of gravitational wave (left) and geon metrics (right). Radial velocity is increasing from left blue to right green curves.

Finally let consider some inflation scenario coming from the state equation (26) in a case of “fermions” and from eq. (27) – Figure 4. For any initial conditions, and for any value of radial velocity there are same exponent grows like in Big Bang model [5].

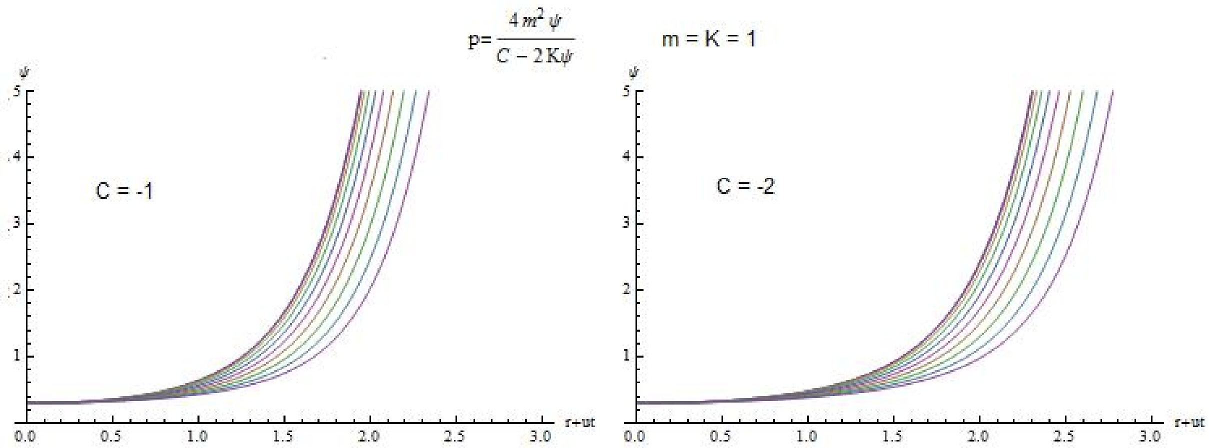

Figure 4: Numerical solutions of eq. (27) in a case of the “fermions” state equation (26). Radial velocity is increasing from right blue to left curves.

Thus, Einstein’s theory of gravitation [1] in the form of equations (5)-(6) with the metric (4) can be used as the basic model to explain the origin of quantum mechanics and quantum statistics in line with the Schrödinger hypothesis [16].

References

1. Einstein A. Zur allgemeinen Relativitatstheorie. Sitzungsber. preuss. Akad. Wiss., 1915, 44, 2, 778—786; Erklarung der Perihelbeivegung der Merkur aus der allgemeinen Relativitdtstheorie. Sitzungsber. preuss. Akad. Wiss., 1915, 47, 2, 831—839; Die Grundlage der allgemeinen Relativitatstneorie. Ann. Phys., 1916, 49, 769—822; Nahemngsweise Integration der Feldgleichungen der Gravitation. Sitzungsber. preuss. Akad. Wiss., 1916, 1, 688—696; Kosmologische Betrachtungen zur allgemeinen Relativitdtstheorie. Sitzungsber. preuss. Akad. Wiss., 1917, 1, 142—152; Uber Gravitationwellen. Sitzungsber. preuss. Akad. Wiss., 1918, 1, 154—167.

2. Einstein A., Infeld L. Gravitational Equations and the Problems of Motion //Ann.Math., 1940,41, 455—464; On the Motion of Particles in General Relativity Theory// Canad. J. Math., 1949, 1, 209—241.
3. Zeldovich, Y. B. The Cosmological Constant and the Theory of Elementary Particles// Soviet Physics Uspekhi vol. 11, 381-393, 1968.
4. Steven Weinberg. The Cosmological Constant Problems// arXiv:astro-ph/0005265v1 12 May 2000.
5. Andrei Linde. Inflationary Cosmology after Planck 2013//arXiv:1402.0526v1 [hep-th] 3 Feb 2014.
6. C.P. Burgess. The Cosmological Constant Problem: Why it's hard to get Dark Energy from Micro-physics//arXiv:1309.4133v1 [hep-th] 16 Sep 2013
7. Trunev A.P. Cosmology of inhomogeneous rotating universe and reality show//, Network electronic scientific journal of the Kuban State Agrarian University (The Journal KubGAU) [electronic resource], Krasnodar KubGAU, 2014. – №01(095). – IDA [article ID]: 0951401028, http://ej.kubagro.ru/2014/01/pdf/28.pdf
8. Trunev A.P. QUANTUM GRAVITY AND YANG-MILLS THEORY // Network electronic scientific journal of the Kuban State Agrarian University (The Journal KubGAU) [electronic resource], Krasnodar KubGAU, 2014. – №01(095) http://ej.kubagro.ru/2014/01/pdf/70.pdf
9. Wheeler J. A. On the Nature of Quantum Geometrodynamics// Annals of Physics 2, No, 6 (Dec 1957): 604 – 614.
10. John A Wheeler. Neutrinos, Gravitation, and Geometry/ In Rendiconti della Scuola internazionale di fisica "Enrico Fermi." Corso XI, by L. A.Radicati. Bologna: Zanichelli, 1960, 67 – 196.
11. Sundance O. Bilson-Thompson, Fotini Markopoulou, Lee Smolin. Quantum gravity and the standard model//arXiv:hep-th/0603022, 21 Apr 2007.
12. Planck Collaboration: Cosmological parameters. – Plank 2013 results, Astronomy & Astrophysics manuscript, March 21, 2013; Planck 2013 results. XXVI. Background geometry and topology of the Universe. Submitted to A&A (2013).
13. Bernard de Wit. Supergravity // arXiv: hep-th/0212245v1 19 Dec 2002.
14. Yang C. N., Mills R. L. Conservation of Isotopic Spin and Isotopic Gauge Invariance// Phys. Rev. 96: 191–195. 1954.
15. H. Fritzsch, M. Gell-Mann, H. Leutwyler. Advantages of the color octet gluon picture// Phys. Lett. B 47 (1973) 365.
16. Erwin Schrödinger. Quantisierung als Eigenwertproblem (Erste Mitteilung)//Annalen der Physik, (4), 79, (1926), 361-376; Quantisierung als Eigenwertproblem (Zweite Mitteilung)//Annalen der Physik, (4), 79, (1926), 489-527; Letter Schrodinger to Einstein, Jul 19, 1939.
17. Stephen L. Adler. Where is quantum theory headed? // arXiv:1401.0896 [quant-ph], 5 Jan 2014; Incorporating gravity into trace dynamics: the induced gravitational action//Class. Quantum Grav. 30, 2013.
18. N.H. Ibragimov. Transformation Groups Applied to Mathematical Physics. – Reidel, Boston, 1984.

19. Darren G. Growdy. General Solution to the 2D Liouville Equation//Int. J. Engng Sci., Vol. 35, No. 2, pp. 141-149, 1997.
20. Nir Cohen, Julia V. Toledo Benavides. Exact solutions of Bratu and Liouville equations// CNMAC 2010, pp. 750-756.
21. Polyakov A.M. Quantum geometry of bosonic strings//Phys. Letter, 103B, 3, 1981.
22. Zamolodchikov A, Zamolodchikov Al. Liouville Field Theory on a Pseudosphere// arxiv: hep-th/0101152v1. 23 Jan, 2001.
23. J. Teschner. Liouville theory revisited// arxiv: hep-th/0104158v3, 9 Nov 2001.
24. Yu Nakayama. Liouville Field Theory// arxiv: hep-th/0402009v7, 10Dec, 2004.
25. L. de Broglie. Recherches sur la theorie des quanta. - Thesis (Paris), 1924.
26. Clinton J. Davisson, Lester H. Germer. Diffraction of Electrons by a Crystal of Nickel// Phys. Rev. 30, 705, 1927; Clinton J. Davisson. The discovery of electron waves. Nobel Lecture, Dec 13, 1937.
27. Steven Weinberg. Gravitation and Cosmology. – John Wiley & Sons, 1972.
28. C. Kiefer. Quantum Gravity. – Clarendon Press, Oxford, 2004.
29. Stephen Boughn, Toni Rothman. Aspect of Gravitation Detection: Graviton Emission and Absorption by Atomic Hydrogen// arxiv: gr-gc/0605052v2 Feb 6, 2008.
30. J. Rosner. Planning the Future of U.S. Particles Physics// arxiv: 1401.6075v1 [hep-ex] 23 Jan 2014.
31. S.W. Hawking, G.F.R. Ellis. The large scale structure of space-time. – Cambridge University Press, 1973.
32. Martin Rees, Remo Ruffini, John A Wheeler. Black holes, gravitational waves and cosmology: an introduction to current research. -New York, Gordon and Breach, Science Publishers, Inc. (Topics in Astrophysics and Space Physics. Volume 10), 1974. 182 p
33. Pauli W. The Connection Between Spin and Statistics// Phys. Rev. 58 (8), 716-722, 1940.